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A new class of two-dimensional noncommutative spaces

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Abstract: We find an infinite number of noncommutative geometries which posses a differential structure. They generalize the two dimensional noncommutative plane, and have infinite dimensional representations. Upon applying generalized coherent states we are able to take the continuum limit, where we recover the punctured plane with non constant Poisson structures.

Keywords: Solitons Monopoles and Instantons, Non-Commutative Geometry, Matrix Models.

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Contents

1. Introduction

Although the noncommutative geometry program [1] holds promise for string theory, quantum gravity and renormalization theory, one dilemma facing it is the scarcity of suitable examples. For the case of two dimensions, there are essentially only two such examples: the noncommutative plane (or torus) and the fuzzy sphere. They are distinguished, in part, by the size of their representations, the former being infinite and the latter finite. The main obstacle in constructing more such systems is the requirement that they posses a differential structure, a necessary first step to writing down noncommutative field theories.[1] In this article, we find new examples of noncommutative geometries which posses a differential structure, and whose continuum limits are two dimensional manifolds. They are associated with algebras of the form

$$[z,\bar{z}] = \Theta\,, \tag{1.1}$$

where Θ is a function of $z\bar{z}$. We find an infinite number of solutions consistent with this anzatz. They include the noncommutative plane (Θ=constant) as a special case. As in that case, the algebras have infinite dimensional representations.

The commutative limit for these new examples is the punctured plane. We deduce this after applying generalized coherent states defined on the complex plane. A classical limit can be defined which preserves the differential structure. Upon taking it we get a singularity in the symplectic two form at the origin. At large distances from the origin the Poisson

[1] On the other hand, more possibilities exist if one doesn't make this demand. See for example [2].

structure approaches a constant. Alternatively, we can say that the full noncommutative theory provides a regularization for such a singularity, and a possible interpretation of our solutions is that they correspond to defects on the noncommutative plane, which go to point singularities in the commutative limit. Analogous phenomena were found for the fuzzy sphere by S. Vaidya [3]. A physical application of the solutions found here could be in the description of vortices in the fractional quantum Hall effect. For this we recall the recent proposal by Susskind for writing the fractional quantum Hall effect in terms of noncommutative Chern-Simons theory [4, 5]. There, vortices were inserted aposteriori on the noncommutative plane. However it may be more natural to start with our noncommutative spaces which already contain vortex-like features. We intend to report on this in a future work.

After a brief review of the exterior derivative for the noncommuting plane in section 2, we construct the new noncommutative spaces in section 3. Two steps are needed in this construction. One is to modify the procedure for taking the exterior derivative from that used on the noncommutative plane. The other is to insure that there is no trivializing map of this algebra, i.e. one that takes it to the noncommutative plane. Two distinct coherent state descriptions and star products are constructed for the new noncommutative spaces in sections 4 and 5. The coherent states in section 4 are the standard ones for the harmonic oscillator and the star product is a familiar one, known as the Voros star product [6]. The latter is a 'equivalent' to the Moyal star product [7] and is thus applicable to the noncommuting plane. When applying it to our noncommutative spaces, it, however, has the disadvantage that the commutative limit is obscure. For this reason we recall alternative procedures for constructing coherent states and their associated star products in section 5. Particularly useful is one developed by us along with G. Alexanian in [8]. Although the star product in this section is more involved than the Voros star product, a closed integral expression for it nevertheless exists. (Alternatively, a systematic derivative expansion can be given, and this is done in appendix A.) The main advantage of this star product is that it is easy to take the commutative limit. We do this in section 6. There we show how the singularity develops in the limit. It can be interpreted as a coordinate singularity on a Kähler manifold $\mathcal{M}$. A procedure for quantizing Kähler manifolds was given a long time ago by Berezin [9]. We prove in appendix B that Berezin's quantization procedure is not the inverse of our classical limiting procedure, and it appears not to preserve the differential calculus. Therefore our new noncommutative spaces could not have been discovered by applying Berezin quantization to $\mathcal{M}$.

2. The noncommutative plane

The algebra $\mathcal{A}_{np}$ of the noncommutative plane is generated by the operator z and its hermitian conjugate $\bar{z}$, satisfying the commutation relation (1.1) where Θ is a nonzero central element. An exterior derivative d can be easily defined for any operator A which is consistent with linearity and the Leibniz rule. For this we introduce an operator Q, and write

$$dA = [Q, A]\,. \tag{2.1}$$

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For dz and $d\bar{z}$ to be hermitian conjugates we need that Q is antihermitian, and in order that $d^2 = 0$, the anticommutator $[Q, Q]_+$ should be in the center of $\mathcal{A}_{np}$. This is the case for

$$Q = \Theta^{-1} z \bar{\chi} - \chi \bar{z} \Theta^{-1}\,, \tag{2.2}$$

where χ and its hermitian conjugate $\bar{\chi}$ are assumed to be Grassmann elements satisfying trivial anticommutation relations

$$[\chi, \chi]_+ = [\bar{\chi}, \chi]_+ = [\bar{\chi}, \bar{\chi}]_+ = 0\,. \tag{2.3}$$

They are also assumed to commute with z and $\bar{z}$. As a result, $[Q, Q]_+ = 2\Theta^{-1}\chi\bar{\chi}$, which commutes with z and $\bar{z}$, and we can identify χ and $\bar{\chi}$ with dz and $d\bar{z}$, respectively.

3. Constructing novel noncommutative spaces

Now we generalize to algebras $\mathcal{A}_{\text{new}}$ where Θ is a hermitian operator depending on z and $\bar{z}$. We shall limit the discussion to functions of $z\bar{z}$. To define the exterior derivative, we again assume the existence of an operator Q such that (2.1) holds. Once again we need that Q is antihermitian, and that the anticommutator $[Q, Q]_+$ should be in the center of the algebra.

As a preliminary step, let us see what happens if we keep the definition Q in (2.2), with χ and $\bar{\chi}$ once again satisfying the trivial anticommutation relations (2.3).[2] Now we can no longer identify χ and $\bar{\chi}$ with dz and $d\bar{z}$, respectively, but rather

$$\begin{aligned} dz &= [Q, z] = (1 - \bar{z}[\Theta^{-1}, z])\,\chi + [\Theta^{-1}, z] z\,\bar{\chi} \\ d\bar{z} &= [Q, \bar{z}] = -\bar{z}[\Theta^{-1}, \bar{z}]\,\chi + (1 + [\Theta^{-1}, \bar{z}] z)\,\bar{\chi}\,. \end{aligned} \tag{3.1}$$

For $[Q, Q]_+$ to be in the center of the algebra the following condition on Θ^{-1} must be satisfied:

$$\begin{aligned} [\bar{z}\Theta^{-1}, \Theta^{-1} z] &= \Theta^{-1} + \Theta^{-1}\bar{z}z\Theta^{-1} - \bar{z}\Theta^{-2} z \\ &= \Theta_0^{-1} = \text{central element}\,. \end{aligned} \tag{3.2}$$

Note that $\bar{z}\Theta^{-1}$ and $\Theta^{-1}z$ generate the algebra of the noncommuting plane, so that any solution we find for $\mathcal{A}_{\text{new}}$ will have to contain $\mathcal{A}_{np}$ as a subalgebra.

Next consider a mapping from the noncommutative plane, generated by raising and lowering operators $\mathbf{a}$ and $\mathbf{a}^\dagger$, with $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. We write the map as[3]

$$z = f(\mathbf{n} + 1)\mathbf{a}\,, \tag{3.3}$$

where $f(\mathbf{n} + 1)$ is a hermitian function of the number operator $\mathbf{n} = \mathbf{a}^\dagger\mathbf{a}$. We denote eigenvectors of the latter by $|n\rangle$, $n = 0, 1, 2, \ldots$, which span the Hilbert space $H^{(0)}$, $\mathbf{n}|n\rangle = n|n\rangle$. Then $\Theta(z\bar{z})$ can also be expressed as a function of $\mathbf{n}$

$$\Theta(z\bar{z}) \equiv \tilde{\Theta}(\mathbf{n}) = (\mathbf{n} + 1) f(\mathbf{n} + 1)^2 - \mathbf{n} f(\mathbf{n})^2 \tag{3.4}$$

[2] We thank P. Prešnajder for making this suggestion.

[3] The same map appears in [2].

and consequently

$$f(n+1)^2 = \frac{\Sigma(n)}{n+1}\,, \qquad \Sigma(n) \equiv \sum_{m=0}^{n} \tilde{\Theta}(m)\,. \tag{3.5}$$

Now examine the condition (3.2). From its vacuum expectation value

$$\tilde{\Theta}(0)^{-1} = \Theta_0^{-1}\,, \tag{3.6}$$

while from expectation value for the first excited state we can write

$$\tilde{\Theta}(1)^{-2} = 2\,\Theta_0^{-1}\Sigma(1)^{-1}\,, \tag{3.7}$$

where we used (3.5). Repeated application of this procedure to the higher excited states gives

$$\tilde{\Theta}(n)^{-2} = (n+1)\,\Theta_0^{-1}\Sigma(n)^{-1} \tag{3.8}$$

and also

$$\tilde{\Theta}(n+1)^{-1} + \Sigma(n)\left(\tilde{\Theta}(n+1)^{-2} - \tilde{\Theta}(n)^{-2}\right) = \Theta_0^{-1}\,. \tag{3.9}$$

Upon solving for $\tilde{\Theta}(n+1)^{-1}$, we get the following recursion relation

$$\begin{aligned}\tilde{\Theta}(n+1)^{-1} &= \frac{1}{2\Sigma(n)}\left\{-1 \pm \sqrt{1+4\Sigma(n)\left(\Theta_0^{-1}+\tilde{\Theta}(n)^{-2}\,\Sigma(n)\right)}\,\right\} \\ &= \frac{\Theta_0\tilde{\Theta}(n)^{-2}}{2(n+1)}\left\{-1 \pm \sqrt{1+4(n+1)(n+2)\Theta_0^{-2}\tilde{\Theta}(n)^2}\right\}.\end{aligned} \tag{3.10}$$

If we choose the plus sign in (3.10) and insert the initial value (3.6) at $n=0$ it is easy to check that $\tilde{\Theta}(n)^{-1} = \Theta_0^{-1}$ *for all* $n \geq 0$. So then the only solution is the trivial one, i.e. the noncommutative plane! If we choose the minus sign in (3.10), $\tilde{\Theta}(n)$ goes rapidly to zero for large n. Starting with the assumption $-\Theta_0\tilde{\Theta}(n)^{-1} \gg n$, (3.10) gives[4]

$$\frac{\tilde{\Theta}(n+1)^{-1}}{\tilde{\Theta}(n)^{-1}} \to \frac{-\Theta_0\tilde{\Theta}(n)^{-1}}{n} \gg 1\,, \qquad \text{as} \quad n \to \infty \tag{3.11}$$

This means that $\tilde{\Theta}(n)^{-1}$ grows faster than an exponential, which validates the starting assumption. Since $\tilde{\Theta}(n)^{-1}$ appears in Q, differentiation becomes ill-defined in the limit of large n. This is unacceptable, so our preliminary attempt at finding nontrivial solutions for $\tilde{\Theta}(n)$ fails.[5]

A successful deformation of the noncommuting plane requires a more involved modification than what was tried above. With this in mind, we follow the two steps given below.

[4]Note that in this case Θ_0 and $\tilde{\Theta}(n)$, $n \geq 1$ have opposite sign.

[5]There is yet another possibility which we shall not consider here, and that is to choose different signs in (3.10) for different values of n. To avoid the previously mentioned problems in the asymptotic region we should require the plus sign as $n \to \infty$. On the other hand, with a minus sign as $n \to \infty$, $\tilde{\Theta}(n)$ starting from a fixed value goes rapidly to zero, and this may be one way of ending up with an 'edge' in the commutative limit [5]. In either case, with switches in sign we should expect to obtain singularities upon taking the continuum limit, and these are not unlike the ones we obtain in section 6.

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Step 1. Is to modify the definition of Q in (2.2). We replace Θ^{-1} in Q by an operator M which is the inverse of Θ everywhere except at a finite number of states. We first look at the case of only one such state, namely $|0\rangle$. Then

$$Q = M(\mathbf{n})z\bar{\chi} - \chi\bar{z}M(\mathbf{n})\,, \qquad M(\mathbf{n}) = \tilde{\Theta}(\mathbf{n})^{-1} - \rho_0\,\Theta_0^{-1}\,|0\rangle\langle 0|\,, \tag{3.12}$$

and (3.2) is replaced by

$$\tilde{\Theta}M^2 + M\bar{z}zM - \bar{z}M^2z = \Theta_0^{-1} = \text{central element}\,. \tag{3.13}$$

The vacuum expectation value now gives

$$M(0)^2 = \Theta_0^{-1}\tilde{\Theta}(0)^{-1} \tag{3.14}$$

and consequently using (3.12)

$$\tilde{\Theta}(0)^{-1} = M(0) + \rho_0\Theta_0^{-1} = \frac{1}{2}\,\Theta_0^{-1}\left(1 + 2\rho_0 + \sqrt{1+4\rho_0}\right). \tag{3.15}$$

From now on, our sign choice is plus to avoid the problems encountered previously in the asymptotic region with the opposite choice. We recover the previous initial value $\tilde{\Theta}(0)^{-1} = \Theta_0^{-1}$ when $\rho_0 = 0$, and consequently the noncommutative plane. For general ρ_0, $\tilde{\Theta}(0)^{-1}$ can have any value greater than or equal to $\Theta_0^{-1}/4$. Since $M(1) = \tilde{\Theta}(1)^{-1}$, (3.7) once again holds and we can then solve for $\tilde{\Theta}(1)^{-1}$

$$\begin{aligned}\tilde{\Theta}(1)^{-1} &= \frac{1}{2}\,\tilde{\Theta}(0)^{-1}\left(\,-1 + \sqrt{1 + 8\,\Theta_0^{-1}\tilde{\Theta}(0)}\,\right)\\ &= \frac{1}{4}\,\Theta_0^{-1}(\,1 + 2\rho_0 + \sqrt{1+4\rho_0}\,)\left(-1 + \sqrt{1 + \frac{4}{\rho_0^2}(1 + 2\rho_0 - \sqrt{1+4\rho_0}\,)}\right)\end{aligned} \tag{3.16}$$

which as expected reduces to Θ_0^{-1} when $\tilde{\Theta}(0)^{-1} = \Theta_0^{-1}$. Moreover, (3.8)–(3.10) for $n \geq 1$ are also still valid, allowing us to generate all $\tilde{\Theta}(n)$. For all $\tilde{\Theta}(0)^{-1} \neq \Theta_0^{-1}$, we obtain nontrivial solutions.

To understand the behavior at large n we can replace (3.9) by the first order differential equation

$$\tilde{\Theta}(n)^{-1} + \;(n+1)\Theta_0\tilde{\Theta}(n)^2\,\frac{d}{dn}\tilde{\Theta}(n)^{-2} = \Theta_0^{-1}\,, \tag{3.17}$$

where we used (3.8). The asymptotic solution for large n is

$$\tilde{\Theta}(n)^{-1} \to \Theta_0^{-1} + \frac{C}{\sqrt{n}} \qquad \text{as} \qquad n \to \infty\,, \tag{3.18}$$

where C is a constant. Thus $\tilde{\Theta}(n)$ is well behaved for $n \to \infty$. It approaches a constant value, so we can say that $\mathcal{A}_{\text{new}}$ approaches a noncommutative plane in this limit.

Even though in the above we have managed to obtain a non-constant solution for $\tilde{\Theta}(n)$, the corresponding algebra generated by z and $\bar{z}$ is *equivalent* to that of the noncommutative plane due to the map (3.3) — *provided it is invertible* — as it only corresponds to a change of variables. On the other hand, if we can insure that the map (3.3) is non invertible, we have an algebra that cannot be mapped back to the noncommutative plane. So

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Step 2 is to make the map $\mathbf{a}, \mathbf{a}^\dagger \rightarrow z, \bar{z}$ non invertible. We can implement this by requiring that $f(n)$ has one or more zeros for some $n \geq 1$. (Note that the value of $f(0)$ is arbitrary.) More specifically, we shall look at the case where $f(1) = f(2) = \cdots = f(n_0) = 0$. We first examine the case $n_0 = 1$. It follows from (3.5) that $\tilde{\Theta}(0)$ vanishes. In (3.15) this corresponds to the limit $\rho_0 \rightarrow \infty$. Substituting $\Sigma(1) = \tilde{\Theta}(1)$ in (3.7), or taking the limit $\rho_0 \rightarrow \infty$ in (3.16), gives

$$\tilde{\Theta}(1) = \frac{1}{2}\, \Theta_0 \,.$$

Then once again (3.10) can be used to generate all $\tilde{\Theta}(n)$, $n \geq 2$, and the behavior for large n is given by (3.18). We note in this case that $M(0)$ is singular, and hence differentiation is ill-defined, at $n = 0$. But the state $|0\rangle$ is not present in the Hilbert space for z and $\bar{z}$, which we call $H^{(1)}$, because $z\,|1\rangle = 0$, i.e. $|1\rangle$ is the lowest weight state in $H^{(1)}$.

Steps 1 and 2 can be repeated in a more general setting to obtain more inequivalent algebras. These algebras are labeled by integer n_0, and are infinite in number. Following step 1 we modify the definition of Q to one where $M(n)$ differs from $\tilde{\Theta}^{-1}(n)$, now for more than one value of n. So we generalize $M(n)$ in (3.12) to

$$M(\mathbf{n}) = \tilde{\Theta}(\mathbf{n})^{-1} - \; \Theta_0^{-1} \sum_{m=0}^{n_0-1} \rho_m \; |m\rangle\langle m| \tag{3.19}$$

for some positive integer n_0. Following step 2 we then take all $\tilde{\Theta}(n)$, $n = 0, 1, 2, \ldots, n_0 - 1$ to zero (or equivalently, all $\rho_n \rightarrow \infty$), producing a non invertible map (3.3).(3.8) now generalizes to

$$M(n)^2 = (n+1)\; \Theta_0^{-1} \Sigma(n)^{-1} \tag{3.20}$$

$\tilde{\Theta}(n_0)$ is then determined by setting $n = n_0$ and using $\Sigma(n_0)^{-1} = \tilde{\Theta}(n_0)^{-1} = M(n_0)$:

$$\tilde{\Theta}(n_0) = \frac{\Theta_0}{n_0 + 1} \,. \tag{3.21}$$

Once again (3.10) can be used to generate all $\tilde{\Theta}(n)$, $n \geq n_0 + 1$. So for example

$$\tilde{\Theta}(n_0 + 1) = \left(1 + \sqrt{\frac{5n_0 + 9}{n_0 + 1}} \right) \frac{\Theta_0}{2(n_0 + 2)} \,, \tag{3.22}$$

and the behavior for large n is given by (3.18). Now we have that $M(n)$, $n = 0, 1, 2, \ldots, n_0 - 1$ are singular in the limit, and differentiation becomes ill-defined for all $0 \leq n \leq n_0 - 1$. But all the corresponding states $|n\rangle$, $n = 0, 1, 2, \ldots, n_0 - 1$ are eliminated from the resulting Hilbert space for z and $\bar{z}$, denoted by $H^{(n_0)}$, since $z\,|n_0\rangle = 0$, and now $|n_0\rangle$ is the lowest weight state in $H^{(n_0)}$.

4. Coherent states and star product — type I

In this section and the next we look at two different coherent state descriptions for the above system and construct the corresponding star products. The different coherent state descriptions in this section and the next have certain advantages and disadvantages. The coherent states of this section, which we denote by $\widetilde{|\alpha\rangle}$, have the advantage of simplicity, and they lead to a well known star product.

We define coherent states $\widetilde{|\alpha\rangle} \in H^{(n_0)}$ for complex α

1. to satisfy a partition of unity
$$1 = \int d\tilde{\mu}(\alpha, \bar{\alpha}) \; \widetilde{|\alpha\rangle}\widetilde{\langle\alpha|} \tag{4.1}$$
for some integration measure $d\tilde{\mu}(\alpha, \bar{\alpha})$,
2. to be of unit norm $\widetilde{\langle\alpha|\alpha\rangle} = 1$ and
3. to form rays with respect to the action of the two-dimensional translation group generated by $\bar{z}\Theta^{-1}$ and $\Theta^{-1}z$.

We first introduce the unitary operators
$$U(\alpha, \bar{\alpha}) = e^{\alpha\bar{z}\Theta^{-1} - \bar{\alpha}\Theta^{-1}z} \,. \tag{4.2}$$

From (3.2) we have the composition rule
$$U(\alpha + \beta, \bar{\alpha} + \bar{\beta}) = e^{-\frac{1}{2}\Theta_0^{-1}(\bar{\alpha}\beta - \bar{\beta}\alpha)} U(\beta, \bar{\beta}) U(\alpha, \bar{\alpha}) \tag{4.3}$$

and so the set $\{U(\beta, \bar{\beta})\}$ forms the projective representation of the translation group. We define the coherent states according to
$$\begin{aligned} \widetilde{|\alpha\rangle} &= U(\alpha, \bar{\alpha})|n_0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2\Theta_0^{-1}} e^{\alpha\bar{z}\Theta^{-1}} e^{-\bar{\alpha}\Theta^{-1}z}|n_0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2\Theta_0^{-1}} e^{\alpha\bar{z}\Theta^{-1}}|n_0\rangle \end{aligned} \tag{4.4}$$

From the set of coherent states $\{\widetilde{|\alpha\rangle}\}$ we thus have orbits in $H^{(n_0)}$ passing through the point $|n_0\rangle$. When acted on by $U(\beta, \bar{\beta})$, these states transform as
$$U(\beta, \bar{\beta})\widetilde{|\alpha\rangle} = e^{\frac{1}{2}\Theta_0^{-1}(\bar{\alpha}\beta - \bar{\beta}\alpha)}\widetilde{|\alpha + \beta\rangle} \,,$$

which follows from (4.3), and so they define rays under the action of the translation group. The coherent states $\widetilde{|\alpha\rangle}$ diagonalize the operator $\Theta^{-1}z$. From (4.4)
$$\Theta_0\Theta^{-1}z\widetilde{|\alpha\rangle} = \alpha\widetilde{|\alpha\rangle} \,. \tag{4.5}$$

We can therefore identify the coordinate α with the expectation value of $\Theta_0\Theta^{-1}z$:
$$\alpha = \Theta_0\widetilde{\langle\alpha|}\Theta^{-1}z\widetilde{|\alpha\rangle} \,.$$

Scalar products are easy to compute in this basis, and they are the essential ingredient for obtaining the measure, and also for constructing the star product. From (4.4)
$$\begin{aligned} \widetilde{\langle\alpha|\beta\rangle} &= \langle n_0|U(\alpha, \bar{\alpha})^\dagger U(\beta, \bar{\beta})|n_0\rangle = \langle n_0|U(-\alpha, -\bar{\alpha})U(\beta, \bar{\beta})|n_0\rangle \\ &= e^{\frac{1}{2}\Theta_0^{-1}(\bar{\alpha}\beta - \bar{\beta}\alpha)}\langle n_0|U(\beta - \alpha, \bar{\beta} - \bar{\alpha})|n_0\rangle = e^{\frac{1}{2}\Theta_0^{-1}(2\bar{\alpha}\beta - |\alpha|^2 - |\beta|^2)} \end{aligned} \tag{4.6}$$

and hence

$$|\widetilde{\langle\alpha|}\widetilde{\beta\rangle}|^2 = e^{-\Theta_0^{-1}|\alpha-\beta|^2} \tag{4.7}$$

which is identical to the result for standard coherent states. It follows that the measure and also the star product are identical to the result for standard coherent states. The former is

$$d\tilde{\mu}(\alpha,\bar{\alpha}) \ = \frac{1}{\pi}\Theta_0^{-1} d\alpha_R \, d\alpha_I \,, \tag{4.8}$$

and from it one gets the partition of unity. (α_R and α_I denote the real and imaginary parts of α.) The latter is known as the Voros star product [6] and it is equivalent to the Moyal star product [7, 10]. It is therefore relevant for the noncommutative plane. Following [9], the covariant symbol $\tilde{\mathcal{A}}(\alpha,\bar{\alpha})$ of an operator A on $H^{(n_0)}$ is defined by $\tilde{\mathcal{A}}(\alpha,\bar{\alpha}) = \widetilde{\langle\alpha|}A\widetilde{|\alpha\rangle}$. The star product $\tilde{\star}$ between any two covariant symbols $\tilde{\mathcal{A}}(\alpha,\bar{\alpha})$ and $\mathcal{B}(\alpha,\bar{\alpha})$ associated with operators A and B is defined to be the covariant symbol of the product of operators:

$$[\tilde{\mathcal{A}}\tilde{\star}\tilde{\mathcal{B}}](\alpha,\bar{\alpha}) \ = \widetilde{\langle\alpha|}AB\widetilde{|\alpha\rangle}$$

and here

$$\tilde{\star} = \exp \ \Theta_0 \ \frac{\overleftarrow{\partial}}{\partial\alpha} \ \frac{\overrightarrow{\partial}}{\partial\bar{\alpha}} \,.$$

Another advantage of this basis comes from the computation of derivatives of covariant symbols. Under infinitesimal translations parameterized by ϵ and $\bar{\epsilon}$

$$\begin{aligned} \tilde{\mathcal{A}}(\alpha,\bar{\alpha}) \ &\to \ \tilde{\mathcal{A}}(\alpha+\epsilon,\bar{\alpha}+\bar{\epsilon}) = \widetilde{\langle\alpha|}U(\epsilon,\bar{\epsilon})^\dagger A U(\epsilon,\bar{\epsilon})\widetilde{|\alpha\rangle} \\ &= \ \tilde{\mathcal{A}}(\alpha,\bar{\alpha}) + \epsilon\widetilde{\langle\alpha|}[A,\bar{z}\Theta^{-1}]\widetilde{|\alpha\rangle} - \bar{\epsilon}\widetilde{\langle\alpha|}[A,\Theta^{-1}z]\widetilde{|\alpha\rangle} \,. \end{aligned} \tag{4.9}$$

Therefore

$$\begin{aligned} \frac{\partial}{\partial\alpha} \ \tilde{\mathcal{A}}(\alpha,\bar{\alpha}) \ &= \ \widetilde{\langle\alpha|}[A,\bar{z}\Theta^{-1}]\widetilde{|\alpha\rangle} \\ \frac{\partial}{\partial\bar{\alpha}} \ \tilde{\mathcal{A}}(\alpha,\bar{\alpha}) \ &= \ -\widetilde{\langle\alpha|}[A,\Theta^{-1}z]\widetilde{|\alpha\rangle} \,. \end{aligned} \tag{4.10}$$

In conclusion, by utilizing the operators $\bar{z}\Theta^{-1}$ and $\Theta^{-1}z$ we have recovered the star product for the noncommuting plane. This is not surprising since the operators $\bar{z}\Theta^{-1}$ and $\Theta^{-1}z$ generate the noncommuting plane. By taking the limit $\Theta_0 \to 0$ we then recover the commutative plane parameterized by α and $\bar{\alpha}$, attached with a constant Poisson structure. But we claim that this is not the appropriate commutative limit of the noncommutative theory generated by operators z and $\bar{z}$. The correct commutative theory should be expressed in terms of coordinates which are the commuting analogues of z and $\bar{z}$, along with a Poisson bracket written in terms of these coordinates. The Poisson bracket should be the classical analogue of the commutator, so we should get a non constant Poisson structure. The appropriate coordinates are the covariant symbols of z and $\bar{z}$, which in our basis are $\widetilde{\langle\alpha|}z\widetilde{|\alpha\rangle}$ and $\widetilde{\langle\alpha|}\bar{z}\widetilde{|\alpha\rangle}$. From (4.5),

$$\widetilde{\langle\alpha|}z\widetilde{|\alpha\rangle} = \alpha\Theta_0^{-1} \ \widetilde{\langle\alpha|}\Theta\widetilde{|\alpha\rangle} \,.$$

However, we don't have an explicit expression for the covariant symbol of Θ in terms of α and $\bar\alpha$, nor consequently z and $\bar z$. Furthermore, the result will be non analytic. These are the main disadvantages of the coherent states $\{\widetilde{|\alpha\rangle}\}$, as without knowing the appropriate change of variables $\alpha,\bar\alpha \to \widetilde{\langle\alpha|}z\widetilde{|\alpha\rangle}, \widetilde{\langle\alpha|}\bar z\widetilde{|\alpha\rangle}$, the commutative limit is obscure.

5. Coherent states and star product — type II

A more appropriate basis of coherent states for understanding the commutative limit would be one that diagonalizes z (or $\bar z$), since then no change of variables is necessary. The general procedure for constructing such coherent states was given in [11, 8], and it will be applied to our algebra in this section. We denote eigenvectors of z by $|\zeta\rangle$ for complex ζ. Unlike with the coherent states of the previous section 4, the resulting star product does not have a simple form (except for the case Θ =constant, where we once again get the Voros star product). Nevertheless, a closed integral expression can be given, which can also be evaluated order by order in a derivative expansion. (See appendix A.)

Like the coherent states of the previous section 4, we require $|\zeta\rangle$ to satisfy 1. the partition of unity (now with some new integration measure $d\mu(\zeta,\bar\zeta)$), along with 2. the unit norm condition. Here we replace condition 3. with the requirement that $|\zeta\rangle$ be an eigenstate of z

$$z|\zeta\rangle = \zeta|\zeta\rangle\,. \tag{5.1}$$

We now give the construction for $|\zeta\rangle$. Analogous to the states of the previous section 4, we can write $|\zeta\rangle$ (up to an overall normalization) by acting on the ground state $|n_0\rangle$ with some operator, though the operator in this case will not be unitary.

From (3.5) we have $z\bar z = \Sigma(\mathbf{n})$, and it follows that

$$[\, z\;,\; \bar z\Sigma(\mathbf{n})^{-1}]\;|\zeta\rangle = (1\;-\;\zeta\bar z\Sigma(\mathbf{n})^{-1})\;|\zeta\rangle$$

and so

$$\begin{aligned} z|\zeta\rangle - \frac{1}{1-\zeta\bar z\Sigma(\mathbf{n})^{-1}}z(1-\zeta\bar z\Sigma(\mathbf{n})^{-1})|\zeta\rangle &= \left[z,\frac{1}{1-\zeta\bar z\Sigma(\mathbf{n})^{-1}}\right](1-\zeta\bar z\Sigma(\mathbf{n})^{-1})|\zeta\rangle \\ &= -\frac{1}{1-\zeta\bar z\Sigma(\mathbf{n})^{-1}}[z,1-\zeta\bar z\Sigma(\mathbf{n})^{-1}]|\zeta\rangle = \zeta|\zeta\rangle. \end{aligned}$$

For this to be consistent with (5.1), z should annihilate $(1\;-\;\zeta\bar z\Sigma(\mathbf{n})^{-1})\;|\zeta\rangle$. The latter must therefore be proportional to the lowest weight state $|n_0\rangle$, and thus

$$(1-\zeta\bar z\Sigma(\mathbf{n})^{-1})\;|\zeta\rangle = \mathcal{N}(|\zeta|^2)^{-1/2}\;|n_0\rangle \tag{5.2}$$

or

$$\mathcal{N}(|\zeta|^2)^{1/2}\;|\zeta\rangle = \frac{1}{1\;-\;\zeta\bar z\Sigma(\mathbf{n})^{-1}}\;|n_0\rangle. \tag{5.3}$$

Upon Taylor expanding, this agrees with the expression for the deformed coherent states of [11, 8]

$$\mathcal{N}(|\zeta|^2)^{1/2}\;|\zeta\rangle = |n_0\rangle\;+\;\sum_{n=1}^{\infty}\frac{\zeta^n\;\;|n+n_0\rangle}{\sqrt{\Sigma(n_0)\Sigma(n_0+1)\cdots\Sigma(n_0+n-1)}}\,. \tag{5.4}$$

Requiring $|\zeta\rangle$ to be of unit norm fixes $\mathcal{N}(|\zeta|^2)$,

$$\mathcal{N}(x) = 1 + \sum_{n=1}^{\infty} \left(\prod_{m=n_0}^{n-1+n_0} \Sigma(m) \right)^{-1} x^n \,. \tag{5.5}$$

Alternatively, we can get another expression for $\mathcal{N}(x)$ by first taking the scalar product of (5.2) with $|n_0\rangle$ to get $\langle n_0|\zeta\rangle = \mathcal{N}(|\zeta|^2)^{-1/2}$, and then with $|\zeta\rangle$ to get

$$\begin{aligned} \mathcal{N}(|\zeta|^2)^{-1} &= 1 \;-\; |\zeta|^2 \, \langle\zeta|\Sigma(\mathbf{n})^{-1}|\zeta\rangle \\ &= \langle\zeta| \; 1 - \bar{z}\Sigma(\mathbf{n})^{-1} z \; |\zeta\rangle \end{aligned} \tag{5.6}$$

$\mathcal{N}(|\zeta|^2)^{-1}$ is then said to be the covariant symbol of the operator $1 - \bar{z}\Sigma(\mathbf{n})^{-1} z$. Another useful relation is

$$\frac{d}{dx} \ln \mathcal{N}(x)|_{x=|\zeta|^2} = \langle\zeta|(\mathbf{n} + 1 - n_0)\Sigma(\mathbf{n})^{-1}|\zeta\rangle \,, \tag{5.7}$$

which can be easily verified from the Taylor expansion for $\mathcal{N}(x)^{-1}$ (c.f. (5.5)), and so $\frac{d}{dx} \ln \mathcal{N}(x)$ is the covariant symbol of the operator $(\mathbf{n} + 1 - n_0)\Sigma(\mathbf{n})^{-1}$.

Using the above coherent states, a new star product $\star$ can be defined between any two covariant symbols $\mathcal{A}(\zeta,\bar{\zeta}) = \langle\zeta|A|\zeta\rangle$ and $\mathcal{B}(\zeta,\bar{\zeta}) = \langle\zeta|B|\zeta\rangle$, where A and B are operators acting on $H^{(n_0)}$. It is such that

$$[\mathcal{A} \star \mathcal{B}](\zeta,\bar{\zeta}) = \langle\zeta|AB|\zeta\rangle \,.$$

For standard coherent states where $\tilde{\Theta}(\mathbf{n}) = \Theta_0 =$ constant, it is once again the Voros star product $\star = \exp\, \Theta_0 \, \frac{\overleftarrow{\partial}}{\partial\zeta} \, \frac{\overrightarrow{\partial}}{\partial\bar{\zeta}}$. More generally, an explicit expression for it using generalized or deformed coherent states on the plane was given in [8], and it depends implicitly on $\mathcal{N}(x)$. By performing a derivative expansion, which we do in appendix A, one obtains the following leading three terms acting on functions of ζ and $\bar{\zeta}$:

$$\star = 1 + \frac{\overleftarrow{\partial}}{\partial\zeta}\, \theta(|\zeta|^2) \frac{\overrightarrow{\partial}}{\partial\bar{\zeta}} + \frac{1}{4}\left[\frac{\overleftarrow{\partial^2}}{\partial\zeta^2} \frac{\overrightarrow{\partial}}{\partial\bar{\zeta}} \theta(|\zeta|^2)^2 \frac{\overrightarrow{\partial}}{\partial\bar{\zeta}} + \frac{\overleftarrow{\partial}}{\partial\zeta} \theta(|\zeta|^2)^2 \, \frac{\overleftarrow{\partial}}{\partial\zeta} \frac{\overrightarrow{\partial^2}}{\partial\bar{\zeta}^2} \right] + \cdots \,, \tag{5.8}$$

where $\theta(|\zeta|^2)$ is the covariant symbol of $\tilde{\Theta}(\mathbf{n})$,

$$\theta(|\zeta|^2) = \langle\zeta|\tilde{\Theta}(\mathbf{n})|\zeta\rangle \,. \tag{5.9}$$

Although we don't have an explicit expression for $\theta(x)$ for our case, we can determine some of its features. At the origin:

$$\theta(0) = \langle n_0|\tilde{\Theta}(\mathbf{n})|n_0\rangle = \tilde{\Theta}(n_0) = \frac{\Theta_0}{n_0 + 1} \,, \tag{5.10}$$

since $|\zeta = 0\rangle = |n_0\rangle$. From (3.18), we also know that $\theta(x)$ approaches a constant Θ_0 as $x \to \infty$, and it is not difficult to determine how fast it approaches this constant. Because $\theta(x)$ varies slowly in this region we may approximate the star product with a finite number of terms. Now take the expectation value of (3.2) with respect to coherent state $|\zeta\rangle$:

$$\langle\zeta|\tilde{\Theta}(\mathbf{n})^{-1}|\zeta\rangle + \langle\zeta|[\tilde{\Theta}(\mathbf{n})^{-1}, \bar{z}] z \tilde{\Theta}(\mathbf{n})^{-1}|\zeta\rangle + \langle\zeta|\bar{z}\tilde{\Theta}(\mathbf{n})^{-1}[z, \tilde{\Theta}(\mathbf{n})^{-1}]|\zeta\rangle = \Theta_0^{-1} \,. \tag{5.11}$$

To lowest order in the star product this gives

$$\theta(|\zeta|^2)^{-1} + \zeta\frac{\partial}{\partial\zeta}\theta(|\zeta|^2)^{-1} + \bar\zeta\frac{\partial}{\partial\bar\zeta}\theta(|\zeta|^2)^{-1} = \Theta_0^{-1} \tag{5.12}$$

or

$$\theta(x)^{-1} + 2x\frac{d}{dx}\theta(x)^{-1} = \Theta_0^{-1} \tag{5.13}$$

and so for large x

$$\theta(x)^{-1} \approx \Theta_0^{-1}\left(1 + \frac{C}{\sqrt{x}}\right), \tag{5.14}$$

where C is the same constant appearing in (3.18). For this we used $\langle\zeta|\mathbf{n}|\zeta\rangle \approx \Theta_0^2|\zeta|^2$, which is valid at leading order.

We can also determine the asymptotic behavior of $\mathcal{N}(x)$ as $x \to \infty$. At lowest order we must recover the result for the noncommuting plane, namely $\mathcal{N}(x) \approx \exp\Theta_0^{-1}x$. To get the next order we make use of (5.6) and (5.7) to write

$$\langle\zeta|(\mathbf{n}+1)\Sigma(\mathbf{n})^{-1}|\zeta\rangle = \left(\frac{d}{dx}\ln\mathcal{N}(x) + \frac{n_0}{x}(1-\mathcal{N}(x)^{-1}\,)\right)\bigg|_{x=|\zeta|^2}. \tag{5.15}$$

From (3.8) the right hand side is the covariant symbol of $\Theta_0\tilde\Theta^{-2}(\mathbf{n})$. At lowest order in the derivative expansion, we can then approximate the right hand side by $\Theta_0\theta(|\zeta|^2)^{-2}$, while on the left hand side we can drop terms that go like $1/x$, so

$$\Theta_0\theta(x)^{-2} \approx \frac{d}{dx}\ln\mathcal{N}(x)\,. \tag{5.16}$$

Substituting the asymptotic solution for $\theta(x)$ then gives

$$\mathcal{N}(x) \approx \exp\Theta_0^{-1}x\left(1 + \frac{4C}{\sqrt{x}}\right). \tag{5.17}$$

6. The commutative limit

We next take the commutative limit in a way that preserves the differential structure. We define it as follows. First re-scale the variables ζ and $\bar\zeta$ by a factor $\frac{1}{\sqrt{\hbar}}$, and then take the limit $\hbar \to 0$. Consequently, the asymptotic behavior of functions such as $\theta(x)$ also corresponds to the commutative limit. Moreover, the order of the derivative expansion of the star product in (5.8) agrees with the order of $\sqrt{\hbar}$, and for small $\hbar$ we may approximate the star product with a finite number of terms. As required, at lowest order in $\hbar$

$$\begin{aligned}[][\mathcal{A}\star\mathcal{B}](\zeta,\bar\zeta) &\to \mathcal{A}(\zeta,\bar\zeta)\;\mathcal{B}(\zeta,\bar\zeta)\\ [\mathcal{A}\star\mathcal{B}-\mathcal{B}\star\mathcal{A}](\zeta,\bar\zeta) &\to i\hbar\,\{\mathcal{A},\mathcal{B}\}(\zeta,\bar\zeta)\,,\end{aligned} \tag{6.1}$$

where the Poisson bracket is defined by

$$\{\mathcal{A},\mathcal{B}\}(\zeta,\bar\zeta) = -i\theta_c(\sqrt{2}|\zeta|)\left(\frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\;\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) - \frac{\partial}{\partial\zeta}\mathcal{B}(\zeta,\bar\zeta)\;\frac{\partial}{\partial\bar\zeta}\mathcal{A}(\zeta,\bar\zeta)\right),$$

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and where $\theta_c(\sqrt{2}|\zeta|) \equiv \lim_{\hbar\to 0}\theta(|\zeta|^2/\hbar)$. If we introduce cartesian coordinates

$$x^1 = \frac{1}{\sqrt{2}}(\zeta + \bar{\zeta})\,, \qquad x^2 = \frac{1}{\sqrt{2}i}(\zeta - \bar{\zeta})\,,$$

then

$$\{\mathcal{A}, \mathcal{B}\}(\vec{x}) = \theta_c(r)\ \epsilon^{ij}\partial_i\mathcal{A}(\vec{x})\ \partial_j\mathcal{B}(\vec{x})\,.$$

Here $\epsilon^{01} = -\epsilon^{10} = 1$, $r = \sqrt{x^i x^i}$ and $\partial_i = \frac{\partial}{\partial x^i}$, and the fundamental Poisson brackets are

$$\{x^i, x^j\} = \theta_c(r)\epsilon^{ij}\,. \tag{6.2}$$

We can define the exterior derivative of any function $\mathcal{A}(\vec{x})$ in a manner analogous to (2.1) in the full noncommutative theory, i.e. by taking the Poisson bracket of $\mathcal{A}(\vec{x})$ with the analogue of the operator Q, call it now $\mathcal{Q}$,[6]

$$d\mathcal{A}(\vec{x}) = \{\mathcal{Q}, \mathcal{A}\}(\vec{x})\,. \tag{6.3}$$

In analogy to (2.2) we define

$$\mathcal{Q} = \theta_c(r)^{-1}\epsilon_{ij}x^i c^j\,, \tag{6.4}$$

where c^i, $i = 1, 2$ are a basis of one forms and are the analogues of χ and $\bar{\chi}$. We assume c^i have zero Poisson bracket with x^i and with themselves. In general, we cannot identify the basis of one forms c^i with dx^i since

$$dx^i \equiv \{Q, x^i\} = c^i + \epsilon_{ij}\epsilon_{k\ell}\theta_c^{-1}\partial_j\theta_c x^k c^\ell\,. \tag{6.5}$$

For $d^2 = 0$, we need that $\{\mathcal{Q}, \mathcal{Q}\}(\vec{x})$ is in the center of the Poisson algebra. If we define directional derivatives $\mathcal{D}_i$, such that

$$d\mathcal{A}(\vec{x}) = [\mathcal{D}_i\mathcal{A}](\vec{x})\ c^i \tag{6.6}$$

then this condition means that $[\mathcal{D}_1, \mathcal{D}_2] = 0$. From (6.4) we get

$$\{\mathcal{Q}, \mathcal{Q}\} = (\theta_c^{-1} + x^k\partial_k\theta_c^{-1})\ \epsilon_{ij}c^i c^j\,.$$

This is in the center of the Poisson algebra for

$$\theta_c^{-1} + x^k\partial_k\theta_c^{-1} = \theta_0^{-1} = \text{constant}\,. \tag{6.7}$$

The rotationally invariant solution to (6.7) is

$$\theta_c(r) = \frac{\theta_0}{1 + r_0/r}\,, \tag{6.8}$$

where r_0 is a constant. We thus obtain a non constant Poisson structure. In comparing with the asymptotic form of $\theta(|\zeta|^2)$ in the full noncommutative theory (c.f. (5.14)), r_0 and θ_0 get identified with $\sqrt{2\hbar}\ C$ and Θ_0, respectively. The noncommutative plane at lowest order is recovered when $r_0 = 0$. For $r_0 \neq 0$, $\theta_c(r)$ vanishes at the origin, unlike $\theta(|\zeta|^2)$ in the

[6]This is similar to the approach followed in [12].

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full noncommutative theory (c.f. (5.10)). From (5.10), we see that to get the classical limit we must take $n_0 \to \infty$, *in addition to* $\hbar \to 0$. Furthermore, since differentiation involves a factor $\theta_c(r)^{-1}$, we must remove the point at the origin from $\mathbb{R}^2$. The commutative limit is therefore a punctured plane, which we denote by $\mathcal{M}$. (If $r_0 < 0$ there is, in addition, a singularity in θ (or a zero in the density) at $-r_0$. In that case we can distinguish the two regions $r < -r_0$ and $r > -r_0$. The former corresponds to a disc with the central origin removed, and the latter is $\mathbb{R}^2$ with a hole.)

Next we examine the orbits generated by the derivatives $\mathcal{D}_i$ on $\mathcal{M}$, and determine an invariant measure. These orbits are the commutative limit of those obtained with the action of the unitary operator $U(\beta,\bar\beta)$ in sec. 4. Infinitesimal motion along the orbit is given by $\delta_\epsilon x^i = \epsilon^j \mathcal{D}_j x^i = \epsilon^j\{\theta_c^{-1}\epsilon_{kj}x^k, x^i\}$, ϵ^i being infinitesimal. From (6.4) and (6.8) we get

$$\delta_\epsilon x^i = \epsilon^i - \frac{r_0}{r+r_0}\left(\epsilon^i - \epsilon\cdot\hat{x}\,\hat{x}^i\right), \qquad \hat{x} = \frac{\vec{x}}{r}. \tag{6.9}$$

The resulting orbits approach a constant direction as $r \to \infty$, and point radially at the origin. It is easy to write an invariant measure $d\mu_I(\vec{x})$ with respect to these orbits. Since $\mathcal{D}_j c^i = 0$, it is just c^1c^2. Using (6.5) we can express $d\mu_I(\vec{x})$ in terms of x^i and dx^i. The jacobian of the transformation $c^i \to dx^i$ is just $\theta_0\theta_c^{-1}$, so

$$d\mu_I(\vec{x}) = \theta_0\theta_c(r)^{-1}\,dx^1dx^2\,. \tag{6.10}$$

Since $\theta_c(r)^{-1}$ appears in the invariant measure we can again argue that the singular point at the origin must be removed from the manifold.

From the invariant measure we can associate a metric ds^2 on $\mathcal{M}$. From the rotational symmetry

$$ds^2 = \gamma(r)\,dx^idx^i = 2\gamma(r)\,d\zeta d\bar\zeta\,, \tag{6.11}$$

with corresponding infinitesimal area element equal to $\gamma(r)\,dx^1dx^2$. We get the metric upon identifying the area element with the invariant measure (6.10),

$$\gamma = \theta_0\theta_c(r)^{-1} = 1 + \frac{r_0}{r}\,. \tag{6.12}$$

Computing the scalar curvature we find

$$R = \frac{\frac{r_0}{r}(2 + r_0/r)}{(1 + r_0/r)^3}\,. \tag{6.13}$$

Since R is well behaved at the origin - it is only a coordinate singularity.

From (6.2), we see that the infinitesimal area element $\gamma(r)\,dx^1dx^2$ (up to a scale factor) is the symplectic two form for the theory. Then by definition $\mathcal{M}$ is a two dimensional homogeneous Kähler manifold. The Kähler potential Γ is defined by

$$\gamma = \frac{\partial}{\partial\zeta}\frac{\partial}{\partial\bar\zeta}\,\Gamma \tag{6.14}$$

for which we find

$$\Gamma = |\zeta|^2 + 2\sqrt{2}r_0|\zeta|\,. \tag{6.15}$$

From (5.17), and the identification of r_0 with $\sqrt{2\hbar}\, C$, we note that the classical limit $\mathcal{N}_c$ of the normalization factor $\mathcal{N}$ is related to Γ by

$$\mathcal{N}_c(\sqrt{2}|\zeta|) = \lim_{\hbar\to 0} \mathcal{N}(|\zeta|^2/\hbar) = \exp\left(\Gamma/\mathtt{h}\right), \tag{6.16}$$

where we define $\mathtt{h} = \hbar\Theta_0$. This relation in fact shows up for Berezin quantization on Kähler manifolds with a bounded domain in $\mathbb{C}^n$ [9, 13]. In comparing Berezin's techniques to the ones we employed [8], we remark that his expression for the star product appears to be the same as ours, i.e. (A.2). To define it one only needs the scalar product and the integration measure. In Berezin quantization, the scalar product can be expressed in terms of normalization function $\mathcal{N}$ in the same way as in our approach (A.4). Unlike us, Berezin gives an explicit expression (B.1) for the integration measure $d\mu_B(\zeta,\bar\zeta)$, and it is written in terms of 'classical' functions θ_c and Γ, as well as $\mathcal{N}$. In appendix B, we show that this integration measure $d\mu_B(\zeta,\bar\zeta)$ is not applicable to our coherent states $|\zeta\rangle$, as it leads to a violation of the partition of unity. So although Berezin's approach and ours coincide at lowest order in $\mathtt{h}$ (or $\hbar$), they are nevertheless distinct quantization procedures. More precisely, the procedure of constructing coherent states $|\zeta\rangle$ and taking the commutative limit, is not the inverse of Berezin's quantization. Therefore applying Berezin's quantization to $\mathcal{M}$ will lead to a different theory from the one we started with in sec. 3, and in all likelihood it will not have the feature that it admits an exterior derivative.

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A. Derivative expansion of the star product

Here we prove (5.8) using the generalized coherent states (5.4). Following [8], given the scalar product $\langle\zeta|\eta\rangle$ and the integration measure $d\mu(\zeta,\bar\zeta)$ satisfying the partition of unity

$$1 = \int d\mu(\zeta,\bar\zeta)|\zeta\rangle\langle\zeta|\,, \tag{A.1}$$

the following closed form expression for the star product between two functions $\mathcal{A}(\zeta,\bar\zeta)$ and $\mathcal{B}(\zeta,\bar\zeta)$ can be given:

$$[\mathcal{A}\star\mathcal{B}](\zeta,\bar\zeta) = \mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta) : \exp\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta) : |\langle\zeta|\eta\rangle|^2 : \exp\,(\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta} : \mathcal{B}(\zeta,\bar\zeta)\,. \tag{A.2}$$

The colons denote an ordered exponential such that the derivatives don't act on the functions appearing in the exponent, i.e.

$$\begin{aligned} : \exp\,(\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta} : \; &= 1 + (\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta} + \frac{1}{2}(\bar\eta-\bar\zeta)^2\frac{\overrightarrow{\partial^2}}{\partial\bar\zeta^2} + \cdots \\ : \exp\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta) : \; &= 1 + \frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta) + \frac{1}{2}\frac{\overleftarrow{\partial^2}}{\partial\zeta^2}(\eta-\zeta)^2 + \cdots\,. \end{aligned} \tag{A.3}$$

The scalar product can be expressed in terms of $\mathcal{N}$ as

$$\langle\eta|\zeta\rangle = \mathcal{N}(|\eta|^2)^{-1/2}\,\mathcal{N}(|\zeta|^2)^{-1/2}\mathcal{N}(\bar\eta\zeta)\,, \tag{A.4}$$

and the measure $d\mu(\eta,\bar\eta)$ can be written in terms of an inverse Mellin integral transform[8], but neither will be needed here. Using the partition of unity (A.1), it is easy to see that at lowest order in the derivative expansion (A.3), $[\mathcal{A}\star\mathcal{B}](\zeta,\bar\zeta)$ reduces to the pointwise product

$$\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle\;\mathcal{B}(\zeta,\bar\zeta) = \mathcal{A}(\zeta,\bar\zeta)\mathcal{B}(\zeta,\bar\zeta)\,, \tag{A.5}$$

while both first order terms vanish:

$$\begin{aligned}
&\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\frac{\overleftarrow{\partial}}{\partial\zeta}(\eta-\zeta)\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle\mathcal{B}(\zeta,\bar\zeta) = \\
&\quad = \frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\big(\langle\zeta|z|\eta\rangle\langle\eta|\zeta\rangle - \langle\zeta|\eta\rangle\langle\eta|z|\zeta\rangle\big)\;\;\mathcal{B}(\zeta,\bar\zeta) \\
&\quad = \frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\big(\langle\zeta|z|\zeta\rangle - \langle\zeta|z|\zeta\rangle\big)\,\mathcal{B}(\zeta,\bar\zeta) = 0
\end{aligned} \tag{A.6}$$

$$\begin{aligned}
&\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle(\bar\eta-\bar\zeta)\frac{\overrightarrow{\partial}}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) = \\
&\quad = \mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\big(\langle\zeta|\eta\rangle\langle\eta|\bar z|\zeta\rangle - \langle\zeta|\bar z|\eta\rangle\langle\eta|\zeta\rangle\big)\;\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) \\
&\quad = \mathcal{A}(\zeta,\bar\zeta)\big(\langle\zeta|\bar z|\zeta\rangle - \langle\zeta|\bar z|\zeta\rangle\big)\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) = 0\,.
\end{aligned} \tag{A.7}$$

The only nonvanishing second order term is

$$\begin{aligned}
&\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\;\frac{\overleftarrow{\partial}}{\partial\zeta}\;|\eta-\zeta|^2\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle\;\frac{\overrightarrow{\partial}}{\partial\bar\zeta}\;\mathcal{B}(\zeta,\bar\zeta) = \\
&\quad = \frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\int d\mu(\eta,\bar\eta)\;\;\big(\langle\zeta|z|\eta\rangle\langle\eta|\bar z|\zeta\rangle + \langle\zeta|\bar z|\eta\rangle\langle\eta|z|\zeta\rangle - \\
&\qquad\qquad - \langle\zeta|\bar z z|\eta\rangle\langle\eta|\zeta\rangle - \langle\zeta|\eta\rangle\langle\eta|\bar z z|\zeta\rangle\big)\;\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) \\
&\quad = \frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\big(\langle\zeta|z\bar z|\zeta\rangle - \langle\zeta|\bar z z|\zeta\rangle\big)\;\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta) \\
&\quad = \frac{\partial}{\partial\zeta}\mathcal{A}(\zeta,\bar\zeta)\theta(|\zeta|^2)\frac{\partial}{\partial\bar\zeta}\mathcal{B}(\zeta,\bar\zeta)
\end{aligned} \tag{A.8}$$

agreeing with (5.8). Applying the same procedure at the next order gives the terms

$$\begin{aligned}
&\frac{1}{2}\int d\mu(\eta,\bar\eta)\;\frac{\overleftarrow{\partial^2}}{\partial\zeta^2}\;|\eta-\zeta|^2(\eta-\zeta)\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle\;\frac{\overrightarrow{\partial}}{\partial\bar\zeta} + \text{complex conjugate} = \\
&\quad = \frac{1}{2}\,\frac{\overleftarrow{\partial^2}}{\partial\zeta^2}\langle\zeta|[z,\tilde\Theta(\mathbf{n})]|\zeta\rangle\;\frac{\overrightarrow{\partial}}{\partial\bar\zeta} + \text{complex conjugate}
\end{aligned} \tag{A.9}$$

acting on $\mathcal{A}(\zeta,\bar\zeta)$ to the right and $\mathcal{B}(\zeta,\bar\zeta)$ to the left. Although these terms involve three derivatives acting on $\mathcal{A}$ and $\mathcal{B}$, overall, it is fourth order in derivatives since

$$\langle\zeta|[z,\tilde\Theta(\mathbf{n})]|\zeta\rangle = \zeta\star\theta(|\zeta|^2) - \theta(|\zeta|^2)\star\zeta \tag{A.10}$$

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which is $\theta(|\zeta|^2)\frac{\partial}{\partial\bar\zeta}\theta(|\zeta|^2)$ to leading order. The result is

$$\frac{1}{2}\,\overleftarrow{\frac{\partial^2}{\partial\zeta^2}}\theta(|\zeta|^2)\frac{\partial}{\partial\bar\zeta}\theta(|\zeta|^2)\,\overrightarrow{\frac{\partial}{\partial\bar\zeta}} + \text{complex conjugate} \tag{A.11}$$

Another fourth order term comes from

$$\begin{aligned}\frac{1}{4}\int d\mu(\eta,\bar\eta)\;\overleftarrow{\frac{\partial^2}{\partial\zeta^2}}\;|\eta-\zeta|^4\langle\zeta|\eta\rangle\langle\eta|\zeta\rangle\;\overrightarrow{\frac{\partial^2}{\partial\bar\zeta^2}} &= \\ = \frac{1}{4}\,\overleftarrow{\frac{\partial^2}{\partial\zeta^2}}\langle\zeta|2\tilde\Theta(\mathbf{n})^2+[\bar z,\tilde\Theta(\mathbf{n})]z+z[\tilde\Theta(\mathbf{n}),\bar z]|\zeta\rangle\,\overrightarrow{\frac{\partial^2}{\partial\bar\zeta^2}}\end{aligned} \tag{A.12}$$

from which we extract the lowest order contribution:

$$\frac{1}{2}\,\overleftarrow{\frac{\partial^2}{\partial\zeta^2}}\;\theta(|\zeta|^2)^2\;\overrightarrow{\frac{\partial^2}{\partial\bar\zeta^2}}\,. \tag{A.13}$$

The fourth order terms (A.11) and (A.13) can be combined to give the result in (5.8).

B. Comparison with Berezin quantization

Here we show that Berezin's integration measure $d\mu_B(\zeta,\bar\zeta)$ violates the partition of unity (A.1) for our coherent states $|\zeta\rangle$. Our proof is by contradiction. $d\mu_B(\zeta,\bar\zeta)$ is defined to be [9, 13]

$$\begin{aligned} d\mu_B(\zeta,\bar\zeta) &= iH(|\zeta|^2)\;d\zeta\wedge d\bar\zeta\,, \\ H(|\zeta|^2) &= -\frac{\alpha(\mathtt{h})}{2\pi}\theta_c(\sqrt{2}|\zeta|)^{-1}\exp\left(-\Gamma/\,\mathtt{h}\right)\mathcal{N}(|\zeta|^2)\,, \end{aligned} \tag{B.1}$$

where $\alpha(\mathtt{h})$ is some normalization constant. For our system θ_c and Γ are given by (6.8) and (6.15), respectively, and $\mathtt{h}=\hbar\Theta_0$. Using (A.4) the partition of unity (A.1) can be written

$$\mathcal{N}(\bar\zeta\lambda) = \int d\mu(\eta,\bar\eta)\frac{\mathcal{N}(\bar\zeta\eta)\;\mathcal{N}(\bar\eta\lambda)}{\mathcal{N}(|\eta|^2)}\,, \tag{B.2}$$

for arbitrary complex coordinates ζ and λ. After substituting (B.1), the integration over the angular variable can be easily performed, leaving

$$I(n) \equiv \int_0^\infty \frac{d\rho\;\rho^{2n+1}\;H(\rho^2)}{\mathcal{N}(\rho^2)} = \frac{1}{4\pi}\begin{cases} 1\,, & n=0 \\ \hbar^n\prod_{m=n_0}^{n-1+n_0}\Sigma(m)\,, & n\geq 1 \end{cases}\,, \tag{B.3}$$

where factors of $\hbar$ came from the rescaling of the coordinates. Substituting in the expression for $H(\rho^2)$ from (B.1), the integral can be expressed in terms of parabolic-cylinder functions $D_n(u)$ [14]

$$I(n) = -\frac{\alpha(\mathtt{h})e^{r_0^2/2}\;(2n)!}{2\pi\Theta_0}\left(\frac{\mathtt{h}}{2}\right)^{n+1}\left((2n+1)\;D_{-2n-2}(u)+\frac{u}{2}\;D_{-2n-1}(u)\right)\,, \tag{B.4}$$

where $u = 2r_0/\sqrt{\mathtt{h}}$. (B.3) for $n = 0$ can be used to fix the normalization constant $\alpha(\mathtt{h})$, yielding

$$\alpha(\mathtt{h}) = -\frac{e^{-r_0^2/2}\ \Theta_0/\mathtt{h}}{D_{-2}(u) + \frac{u}{2}\ D_{-1}(u)}\,. \tag{B.5}$$

Substituting this result for $\alpha(\mathtt{h})$ in (B.3) for $n \geq 1$ gives

$$(2n)!\left(\frac{\Theta_0}{2}\right)^n \frac{(2n+1)\ D_{-2n-2}(u) + \frac{u}{2}\ D_{-2n-1}(u)}{D_{-2}(u) + \frac{u}{2}\ D_{-1}(u)} = \prod_{m=n_0}^{n-1+n_0} \Sigma(m)\,. \tag{B.6}$$

This equality works for the case of the noncommutative plane $r_0 = u = 0$. In this limit, $D_{-2n-2}(0) = 2^n n!/(2n+1)!$, and both the left and right hand sides reduce to $n!\ \Theta_0^n$. For the case of nonzero r_0, we can use one of the conditions (B.6) to determine the relationship between r_0 and n_0. Take for example $n = 1$. Then using (3.21),

$$\frac{3\ D_{-4}(u) + \frac{u}{2}\ D_{-3}(u)}{D_{-2}(u) + \frac{u}{2}\ D_{-1}(u)} = \frac{1}{n_0 + 1}\,. \tag{B.7}$$

Solving this in general and substituting back into (B.6) for $n \geq 2$ is nontrivial. However (B.6) should hold for all n_0, and to disprove it only requires a counter example. For this purpose we look at the case of large n_0. This means we are approaching the commutative limit where u must also be large. Here we can use the following asymptotic expansion for the parabolic-cylinder functions

$$D_p(u) = e^{-u^2/4} u^p (1 + O(u^2))\,. \tag{B.8}$$

At leading order we get $u \to \sqrt{n_0}$ or $r_0 \to \frac{1}{2}\sqrt{\hbar n_0 \Theta_0}$. Substituting this result into the left hand side of (B.6) for $n = 2$ gives $6(\Theta_0/n_0)^2$ to leading order. Using (3.21) and (3.22), the right hand side of (B.6) for $n = 2$ instead gives $(\Theta_0/n_0)^2(\sqrt{5}+1)/(\sqrt{5}-1)$. So the relation (B.6) fails and Berezin's integration measure $d\mu_B(\zeta,\bar{\zeta})$ is not ours.

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